

Poisson multi-Bernoulli conjugate prior for multiple extended object estimation

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Abstract—This paper presents a Poisson multi-Bernoulli mixture (PMBM) conjugate prior for multiple extended object filtering. A Poisson point process is used to describe the existence of yet undetected targets, while a multi-Bernoulli mixture describes the distribution of the targets that have been detected. The prediction and update equations are presented for the standard transition density and measurement likelihood. Both the prediction and the update preserve the PMBM form of the density, and in this sense the PMBM density is a conjugate prior. However, the unknown data associations lead to an intractably large number of terms in the PMBM density, and approximations are necessary for tractability. A gamma Gaussian inverse Wishart implementation is presented, along with methods to handle the data association problem. A simulation study shows that the extended target PMBM filter outperforms the extended target δ -GLMB and LMB filters. An experiment with Lidar data illustrates the benefit of tracking both detected and undetected targets.

I. INTRODUCTION

Multiple target tracking (MTT) is the processing of sets of measurements obtained from multiple sources in order to maintain estimates of the targets' current states¹. Solving the MTT problem is complicated by the fact that—in addition to noise, missed detections and clutter—the number of targets is unknown and time-varying. Point target MTT is defined as tracking targets that give rise to at most one measurement per target at each time step, and extended target MTT is defined as tracking targets that potentially give rise to more than one measurement at each time step, where the set of measurements are spatially distributed around the extended target.

The focus of this paper is on extended targets. A target may give rise to more than one measurement if the resolution of the sensor, the size of the target, and the distance between target and sensor, are such that multiple resolution cells of the sensor are occupied by a single target. Examples of such scenarios include vehicle tracking using automotive radars, tracking of ships with marine radar stations, and person tracking using laser range sensors. An introduction to extended target tracking and a comprehensive overview of the literature is given in [1].

A common extended target measurement model is the inhomogeneous Poisson Point Process (PPP), proposed in [2]. At each time step, a Poisson distributed random number of

measurements are generated, distributed around the target. For tracking multiple extended targets, random finite sets (RFSs) can be used to model the problem. RFSs and Finite Set Statistics (FISST) [3], [4] is a theoretically elegant and appealing approach to the MTT problem where targets and measurements are modelled as random sets. The PPP extended target model [2] has been integrated into several computationally feasible RFS-based filters, e.g., [5]–[10].

In Bayesian probability theory, the concepts of *conjugacy* and *conjugate prior*, first introduced by Raiffa and Schlaifer [11], are important. Conjugacy in the context of MTT means that “if we start with the proposed conjugate initial prior, then all subsequent predicted and posterior distributions have the same form as the initial prior” [12, p. 3460]. MTT conjugate priors are of great interest as they provide families of distributions that are suitable to work with when we seek accurate approximations to the true posterior distributions.

In this paper, we derive a Poisson multi-Bernoulli mixture (PMBM) MTT conjugate prior for the standard extended target models. The PMBM conjugate prior, proposed for point target tracking in [13], allows an elegant separation of the set of targets into two disjoint subsets: targets that have been detected, and targets that have not yet been detected. A preliminary version of this work was presented in [14]. This paper is a significant extension of that work, and contains the following contributions:

- 1) In Section IV, we present the conjugate prediction and the conjugate update for the PMBM density.
- 2) In Section V, we analyse the complexity of the PMBM filter, discuss how the data association problem can be handled, and analyse the approximation error that is incurred by approximating the data association. In Section V-C, we propose a merging algorithm that can be used to reduce the number of components in a multi Bernoulli mixture.
- 3) In Section VI, we present a computationally feasible implementation of the PMBM filter, based on gamma Gaussian inverse Wishart (GGIW) single target models.
- 4) In Section VII, we present a simulation study, where the GGIW-PMBM filter is compared to state-of-the-art algorithms, and we present an experiment, in which the benefits of modelling the targets that have not yet been detected is highlighted.

Problem formulation and modelling are presented in Section II and Section III, respectively. The paper is concluded in Section VIII.

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¹The ultimate output from an MTT algorithm is a set of target trajectories, i.e., a set of target state sequences. In MTT, a multi-target *tracker* produces estimates of target trajectories, while a multi-target *filter* produces estimates of the current set of targets. In this paper, we focus on filtering, whereas forming target trajectories is outside the scope of the paper.

II. PROBLEM FORMULATION

The set of targets at time step k is denoted \mathbf{X}_k , and is modelled as a RFS, meaning that the target set cardinality $|\mathbf{X}_k|$ is a time-varying discrete random variable, and each target state is a random variable. The target state models both kinematic properties (position, velocity, turn-rate, orientation, etc) and target extent (shape and size). The set of measurements at time step k is an RFS denoted \mathbf{Z}_k . There are two types of measurements: clutter measurements and target originated measurements, and the measurement origin is assumed unknown. Further, \mathbf{Z}^k denotes all measurement sets \mathbf{Z}_t from time $t = 0$ up to, and including, time $t = k$.

The multi-object posterior density at time k , given all measurement sets up to and including time step k , is denoted $f_{k|k}(\mathbf{X}_k|\mathbf{Z}^k)$. The multitarget Bayes filter propagates in time the multi-target set density $f_{k-1|k-1}(\mathbf{X}_{k-1}|\mathbf{Z}^{k-1})$ using the Chapman-Kolmogorov prediction

$$f_{k|k-1}(\mathbf{X}_k|\mathbf{Z}^{k-1}) = \int f_{k,k-1}(\mathbf{X}_k|\mathbf{X}_{k-1})f_{k-1|k-1}(\mathbf{X}_{k-1}|\mathbf{Z}^{k-1})\delta\mathbf{X}_{k-1}, \quad (1a)$$

and then updates the density using the Bayes update

$$f_{k|k}(\mathbf{X}_k|\mathbf{Z}^k) = \frac{f_k(\mathbf{Z}_k|\mathbf{X}_k)f_{k|k-1}(\mathbf{X}_k|\mathbf{Z}^{k-1})}{\int f_k(\mathbf{Z}_k|\mathbf{X}_k)f_{k|k-1}(\mathbf{X}_k|\mathbf{Z}^{k-1})\delta\mathbf{X}_k}, \quad (1b)$$

where $f_{k+1,k}(\mathbf{X}_{k+1}|\mathbf{X}_k)$ is the multi-object transition density, $f_k(\mathbf{Z}_k|\mathbf{X}_k)$ is the multitarget measurement set density, and the integrals are set-integrals, defined in [3, Sec. 11.3.3]. In this paper, we model the measurement set density $f_k(\mathbf{Z}_k|\mathbf{X}_k)$ using the standard PPP extended target measurement model [2] and a standard PPP clutter model. The multi-object transition density $f_{k+1,k}(\mathbf{X}_{k+1}|\mathbf{X}_k)$ is modeled by a standard multi-object Markov density with PPP birth.

Among RFS based filters, there are two main filter types that implement the Bayes recursion (1) for the multi-object density. The first is based on moment approximations, e.g., the PHD filter and the CPHD filter. The second is based on parameterised density representations, e.g., the GLMB filter and the PMBM filter.

The main objectives of this paper are: 1) to show that the PMBM representation of $f_{k|k}(\mathbf{X}_k|\mathbf{Z}^k)$ is an MTT conjugate prior for the standard extended target tracking models by deriving the corresponding prediction and update, 2) to show how this PMBM filter can be implemented in a computationally tractable way, and 3) to evaluate the performance of the implementation and compare to state-of-the-art algorithms.

III. MODELING

An introduction to RFSS is given in, e.g., [3], and an introduction to extended object modelling is given in, e.g., [1]. This section first presents a review of random set theory; specifically the PPP and the Bernoulli process. The standard extended target measurement and motion models are then presented. Notation is given in Table I.

TABLE I
NOTATION

- Minor non-bold letters, e.g., α, β, γ , denote scalars, minor bold letters, e.g., $\mathbf{x}, \mathbf{z}, \boldsymbol{\xi}$, denote vectors, capital non-bold letters, e.g., X, H, F denote matrices, and capital bold letters, e.g., $\mathbf{X}, \mathbf{Z}, \mathbf{C}$, denote sets.
- $|V|$: determinant of matrix V .
- $|\mathbf{X}|$: cardinality of set \mathbf{X} , i.e., number of elements in set \mathbf{X} .
- \uplus denotes disjoint set union, i.e., $\mathbf{Y} \uplus \mathbf{Z} = \mathbf{X}$ means that $\mathbf{Y} \cup \mathbf{Z} = \mathbf{X}$ and $\mathbf{Y} \cap \mathbf{Z} = \emptyset$.
- \mathbf{I}_m : identity matrix of size $m \times m$.
- $\langle a; b \rangle = \int a(x)b(x)dx$: inner product of $a(x)$ and $b(x)$.
- $h^{\mathbf{X}} = \prod_{\mathbf{x} \in \mathbf{X}} h(\mathbf{x})$, where $h^{\emptyset} = 1$ by definition.

A. Review of random set modeling

1) *Poisson point process*: A PPP is a type of RFS whose cardinality is Poisson distributed, and all elements (e.g., target states) are independent and identically distributed (iid). A PPP can be parameterised by an intensity function $D(\mathbf{x})$, defined on single target state space. The intensity function can be broken down into two parts $D(\mathbf{x}) = \mu f(\mathbf{x})$: the scalar Poisson rate $\mu > 0$ and the spatial distribution $f(\mathbf{x})$. One important property of the intensity is that $\int_{\mathbf{x} \in S} D(\mathbf{x})d\mathbf{x}$ is the expected number of set members in S . This can be interpreted to mean that in parts of the state space with high/low intensity $D(\mathbf{x})$, there is a high/low chance that set members are located. The PPP density is

$$f(\mathbf{X}) = e^{-\langle D; 1 \rangle} \prod_{\mathbf{x} \in \mathbf{X}} D(\mathbf{x}) = e^{-\mu} \prod_{\mathbf{x} \in \mathbf{X}} \mu f(\mathbf{x}). \quad (2)$$

In this work, PPPs are used to model clutter measurements, extended target measurements, target birth, and undetected targets.

2) *Bernoulli process*: A Bernoulli RFS \mathbf{X} is a type of RFS that is empty with probability $1 - r$ or, with probability r , contains a single element with pdf $f(\mathbf{x})$. The cardinality is therefore Bernoulli distributed with parameter $r \in [0, 1]$. The Bernoulli density is

$$f(\mathbf{X}) = \begin{cases} 1 - r & \mathbf{X} = \emptyset \\ rf(\mathbf{x}) & \mathbf{X} = \{\mathbf{x}\} \\ 0 & |\mathbf{X}| \geq 2 \end{cases} \quad (3)$$

In MTT, a Bernoulli RFS is a natural representation of a single target, as it captures both the uncertainty regarding the target's existence (via the parameter r), as well as the uncertainty regarding the target's state \mathbf{x} (via the density $f(\mathbf{x})$).

A typical assumption in MTT is that the targets are independent, see, e.g., [15]. A multi Bernoulli (MB) RFS \mathbf{X} is the disjoint union of a fixed number of independent Bernoulli RFSS \mathbf{X}^i , $\mathbf{X} = \uplus_{i \in \mathbb{I}} \mathbf{X}^i$, where \mathbb{I} is an index set. The MB density for a set \mathbf{X} can be expressed as

$$f(\mathbf{X}) = \begin{cases} \sum_{\uplus_{i \in \mathbb{I}} \mathbf{X}^i = \mathbf{X}} \prod_{i \in \mathbb{I}} f^i(\mathbf{X}^i) & \text{if } |\mathbf{X}| \leq |\mathbb{I}|, \\ 0 & \text{if } |\mathbf{X}| > |\mathbb{I}|. \end{cases} \quad (4)$$

The MB distribution is defined entirely by the parameters $\{r^i, f^i\}_{i \in \mathbb{I}}$ of the involved Bernoulli RFSS.

Lastly, an MB mixture (MBM) density is an RFS density that is a normalized, weighted sum of MB densities. In MTT the weights typically correspond to the probability of different data association sequences. An MBM is defined entirely by the set

of parameters $\{(\mathcal{W}^j, \{r^{j,i}, f^{j,i}\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}$, where \mathbb{J} is an index set for the MBs in the MBM (also called components of the MBM), \mathbb{I}^j is an index set for the Bernoullis in the j th MB, and \mathcal{W}^j is the probability of the j th MB.

B. Standard extended target measurement model

The set of measurements \mathbf{Z}_k is the union of a set of clutter measurements and a set of target generated measurements; the sets are assumed independent. The clutter is modelled as a PPP with intensity $\kappa(\mathbf{z}) = \lambda c(\mathbf{z})$. An extended target with state \mathbf{x} is detected with state dependent probability of detection $p_D(\mathbf{x})$, and, if it is detected, the target measurements are modelled as a PPP with intensity $\gamma(\mathbf{x})\phi(\mathbf{z}|\mathbf{x})$, where both the Poisson rate $\gamma(\mathbf{x})$ and the spatial distribution $\phi(\mathbf{z}|\mathbf{x})$ are state dependent. A PPP with a probability of detection is sometimes called zero-inflated PPP.

For a non-empty set of measurements ($|\mathbf{Z}| > 0$), the conditional extended target measurement set likelihood is the product of the probability of detection and the PPP density,

$$\ell_{\mathbf{Z}}(\mathbf{x}) = p_D(\mathbf{x})p(\mathbf{Z}|\mathbf{x}) = p_D(\mathbf{x})e^{-\gamma(\mathbf{x})} \prod_{\mathbf{z} \in \mathbf{Z}} \gamma(\mathbf{x})\phi(\mathbf{z}|\mathbf{x}). \quad (5)$$

The effective probability of detection for an extended target with state \mathbf{x} is $p_D(\mathbf{x})(1 - e^{-\gamma(\mathbf{x})})$, where $1 - e^{-\gamma(\mathbf{x})}$ is the Poisson probability of generating at least one detection. Accordingly, the effective probability of missed detection, i.e., the probability that the target is not detected, is

$$q_D(\mathbf{x}) = 1 - p_D(\mathbf{x}) + p_D(\mathbf{x})e^{-\gamma(\mathbf{x})}. \quad (6)$$

Note that $q_D(\mathbf{x})$ is the conditional likelihood for an empty set of measurements, i.e., $\ell_{\emptyset}(\mathbf{x}) = q_D(\mathbf{x})$ (cf. (5)).

Because of the unknown measurement origin², it is necessary to discuss data association. Let the measurements in the set \mathbf{Z} be indexed by $m \in \mathbb{M}$,

$$\mathbf{Z} = \{\mathbf{z}^m\}_{m \in \mathbb{M}} \quad (7)$$

and let \mathcal{A}^j be the space of all data associations A for the j th predicted global hypothesis, i.e., the j th predicted MB. A data association $A \in \mathcal{A}^j$ is an assignment of each measurement in \mathbf{Z} to a source, either to the *background* (clutter or new target) or to one of the existing targets indexed by \mathbb{I}^j . Note that $\mathbb{M} \cap \mathbb{I}^j = \emptyset$ for all j .

The space of all data associations for the j th hypothesis is $\mathcal{A}^j = \mathcal{P}(\mathbb{M} \cup \mathbb{I}^j)$, i.e., a data association $A \in \mathcal{A}^j$ is a partition of $\mathbb{M} \cup \mathbb{I}^j$ into non-empty disjoint subsets $C \in A$, called index cells³. Due to the standard MTT assumption that the targets generate measurements independent of each other, an index cell contains at most one target index, i.e., $|C \cap \mathbb{I}^j| \leq 1$ for all $C \in A$. Any association in which there is at least one cell, with

²An inherent property of MTT is that it is unknown which measurements are from targets and which are clutter, and among the target generated measurements it is unknown which target generated which measurement. Hence, the update must handle this uncertainty.

³For example, let $\mathbb{M} = (m_1, m_2, m_3)$ and $\mathbb{I} = (i_1, i_2)$, i.e., three measurements and two targets. One valid partition of $\mathbb{M} \cap \mathbb{I}$, i.e., one of the possible associations, is $\{m_1, m_2, i_1\}, \{m_3\}, \{i_2\}$. The meaning of this is that measurements m_1, m_2 are associated to target i_1 , target i_2 is not detected, and measurement m_3 is not associated to any previously detected target, i.e., measurement m_3 is either clutter or from a new target.

at least two target indices, will have zero likelihood because this violates the independence assumption. If the index cell C contains a target index, then let i_C denote the corresponding target index. Further, let \mathbf{C}_C denote the measurement cell that corresponds to the index cell C , i.e., the set of measurements

$$\mathbf{C}_C = \bigcup_{m \in C \cap \mathbb{M}} \mathbf{z}^m. \quad (8)$$

C. Standard dynamic model

The existing targets—both the detected and the undetected—survive from time step k to time step $k+1$ with state dependent probability of survival $p_S(\mathbf{x}_k)$. The targets evolve independently according to a Markov process with transition density $f_{k+1,k}(\mathbf{x}_{k+1}|\mathbf{x}_k)$. New targets appear independently of the targets that already exist. The target birth is assumed to be a PPP with intensity $D_{k+1}^b(\mathbf{x})$. In this work, target spawning is omitted; for work on spawning in an extended target context see [16].

IV. POISSON MULTI-BERNOULLI MIXTURE FILTER

In this section, the PMBM conjugate prior for the standard extended object measurement and motion models are presented. Throughout the section time indexing is omitted for the sake of brevity.

A. PMBM density

The PMBM model is a combination of a PPP and a MBM, where the PPP describes the distribution of the targets that are thus far undetected, and the MBM describes the distribution of the targets that have been detected at least once. Thus, the set of targets can be divided into two disjoint subsets,

$$\mathbf{X} = \mathbf{X}^u \uplus \mathbf{X}^d, \quad (9)$$

corresponding to undetected targets \mathbf{X}^u , and detected targets \mathbf{X}^d . The PMBM set density can be expressed as

$$f(\mathbf{X}) = \sum_{\mathbf{X}^u \uplus \mathbf{X}^d = \mathbf{X}} f^u(\mathbf{X}^u) \sum_{j \in \mathbb{J}} \mathcal{W}^j f^j(\mathbf{X}^d), \quad (10a)$$

$$f^u(\mathbf{X}^u) = e^{-\langle D^u; 1 \rangle} \prod_{\mathbf{x} \in \mathbf{X}^u} D^u(\mathbf{x}), \quad (10b)$$

$$f^j(\mathbf{X}^d) = \sum_{\uplus_{i \in \mathbb{I}^j} \mathbf{x}^i = \mathbf{X}^d} \prod_{i \in \mathbb{I}^j} f^{j,i}(\mathbf{x}^i), \quad (10c)$$

where $f^{j,i}(\cdot)$ are Bernoulli set densities, defined in (3). There are $|\mathbb{J}|$ components in the MB mixture, the j th component has $|\mathbb{I}^j|$ Bernoulli components, and the probability of the j th MB component is \mathcal{W}^j . In target tracking each of the MB components in the mixture corresponds to a unique *global hypothesis* for the detected targets, i.e., a particular history of data associations for all detected targets.

The PMBM density is defined entirely by the parameters

$$D^u, \{(\mathcal{W}^j, \{r^{j,i}, f^{j,i}\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}. \quad (11)$$

Since the PMBM density is a MTT conjugate prior, performing prediction and update means that we compute the new PMBM density parameters.

B. PMBM filter recursion

The PMBM filter consist of a prediction and an update step. The PMBM conjugate prediction is presented in Theorem 1.

Theorem 1: Given a posterior PMBM density with parameters

$$D^u, \{(\mathcal{W}^j, \{(r_+^{j,i}, f_+^{j,i})\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}, \quad (12)$$

and the standard dynamic model (Section III-C), the predicted density is a PMBM density with parameters

$$D_+^u, \{(\mathcal{W}_+^j, \{(r_+^{j,i}, f_+^{j,i})\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}, \quad (13)$$

where

$$D_+^u(\mathbf{x}) = D^b(\mathbf{x}) + \langle D^u; p_S f_{k+1,k} \rangle, \quad (14a)$$

$$r_+^{j,i} = \langle f_+^{j,i}; p_S \rangle r_+^{j,i}, \quad (14b)$$

$$f_+^{j,i}(\mathbf{x}) = \frac{\langle f_+^{j,i}; p_S f_{k+1,k} \rangle}{\langle f_+^{j,i}; p_S \rangle}, \quad (14c)$$

and $\mathcal{W}_+^j = \mathcal{W}^j$. \square

The proof of the theorem is omitted for brevity, details can be found in, e.g., [13]. The PMBM conjugate update is presented in Theorem 2.

Theorem 2: Given a prior PMBM density with parameters

$$D_+^u, \{(\mathcal{W}_+^j, \{(r_+^{j,i}, f_+^{j,i})\}_{i \in \mathbb{I}_+^j})\}_{j \in \mathbb{J}_+}, \quad (15)$$

a set of measurements \mathbf{Z} , and the standard measurement model (Section III-B), the updated density is a PMBM density

$$f(\mathbf{X}|\mathbf{Z}) = \sum_{\mathbf{X}^u \oplus \mathbf{X}^d = \mathbf{X}} f^u(\mathbf{X}^u) \sum_{j \in \mathbb{J}_+} \sum_{A \in \mathcal{A}^j} \mathcal{W}_A^j f_A^j(\mathbf{X}^d), \quad (16a)$$

$$f^u(\mathbf{X}^u) = e^{-\langle D^u; 1 \rangle} \prod_{\mathbf{x} \in \mathbf{X}^u} D^u(\mathbf{x}), \quad (16b)$$

$$f_A^j(\mathbf{X}^d) = \sum_{\substack{\mathbf{C} \in \mathcal{A} \\ \mathbf{C} \cap A = \emptyset}} \prod_{C \in \mathbf{C}} f_C^j(\mathbf{X}^d), \quad (16c)$$

where the weights are

$$\mathcal{W}_A^j = \frac{\mathcal{W}_+^j \prod_{C \in A} \mathcal{L}_C}{\sum_{j' \in \mathbb{J}} \sum_{A' \in \mathcal{A}^{j'}} \mathcal{W}_+^{j'} \prod_{C' \in A'} \mathcal{L}_{C'}}, \quad (17a)$$

$$\mathcal{L}_C = \begin{cases} \kappa^{\mathbf{C}_C} + \langle D_+^u; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}^j = \emptyset, |\mathbf{C}_C| = 1, \\ \langle D_+^u; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}^j = \emptyset, |\mathbf{C}_C| > 1, \\ 1 - r_+^{j,i_C} + r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C = \emptyset, \\ r_+^{j,i_C} \langle f_+^{j,i_C}; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C \neq \emptyset, \end{cases} \quad (17b)$$

the densities $f_C^j(\mathbf{X})$ are Bernoulli densities with parameters

$$r_C^j = \begin{cases} \frac{\langle D_+^u; \ell_{\mathbf{C}_C} \rangle}{\kappa^{\mathbf{C}_C} + \langle D_+^u; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}^j = \emptyset, |\mathbf{C}_C| = 1, \\ 1 & \text{if } C \cap \mathbb{I}^j = \emptyset, |\mathbf{C}_C| > 1, \\ \frac{r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle}{1 - r_+^{j,i_C} + r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle} & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C = \emptyset, \\ 1 & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C \neq \emptyset, \end{cases} \quad (17c)$$

$$f_C^j(\mathbf{x}) = \begin{cases} \frac{\ell_{\mathbf{C}_C}(\mathbf{x}) D_+^u(\mathbf{x})}{\langle D_+^u; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}^j = \emptyset, \\ \frac{q_D(\mathbf{x}) f_+^{j,i_C}(\mathbf{x})}{\langle f_+^{j,i_C}; q_D \rangle} & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C = \emptyset, \\ \frac{\ell_{\mathbf{C}_C}(\mathbf{x}) f_+^{j,i_C}(\mathbf{x})}{\langle f_+^{j,i_C}; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}^j \neq \emptyset, \mathbf{C}_C \neq \emptyset, \end{cases} \quad (17d)$$

and the updated PPP intensity is $D^u(\mathbf{x}) = q_D(\mathbf{x}) D_+^u(\mathbf{x})$. \square

The proof of the theorem can be found in Appendix A. By comparing (16) with the PMBM density (10), we can immediately identify that we have a PMBM density. The number of components in the MBM increases, and contains one MB for every pair of predicted MB, $j \in \mathbb{J}_+$, and possible association, $A \in \mathcal{A}^j$.

V. COMPLEXITY, REDUCTION, AND APPROXIMATION ERROR

Due to the unknown number of data associations, the number of components in the MBM grows rapidly as more data is observed, and it follows that the number of PMBM parameters increases. In this section we give an expression for the number of MBM components and illustrate this for a simple example. We then discuss reduction methods that can be used to keep the number of MBM components at a tractable level, and lastly we discuss the approximation error that this reduction incurs.

A. Complexity analysis

Let the filter be initialised with $\mathbb{J}_0 = \{j_1\}$, $\mathcal{W}_0^{j_1} = 1$, and $\mathbb{J}_0^1 = \emptyset$, let the birth intensity $D^b(\mathbf{x}) > 0$ and/or the initial undetected intensity $D^u(\mathbf{x}) > 0$, and let the probabilities of survival and detection be non-zero, $p_S(\mathbf{x}) \in (0, 1]$ and $p_D(\mathbf{x}) \in (0, 1]$, respectively. Then, the number of MBM components, given measurement sets up to and including time step k , is given by the Bell number, denoted $B(n)$, whose order n is the sum of the measurement set cardinalities,

$$|\mathbb{J}_k| = |\mathbb{J}_{k+1}| = B\left(\sum_{t=1}^k |\mathbf{Z}_t|\right). \quad (18)$$

This is the same as the number of ways that we can partition $\bigcup_{t=1}^k \mathbf{Z}_t$ [17]. The sequence of Bell numbers $B(n)$ is log-convex⁴, and $B(n)$ grows very rapidly. For example, for two measurements sets \mathbf{Z}_1 and \mathbf{Z}_2 , both with two measurements, there are $B(2+2) = 15$ hypotheses. A small increase in the number of detections per time step to four (twice the amount), results in an MBM with $B(4+4) = 4140$ hypotheses.

Each MB component in the MBM corresponds to a unique global hypothesis, where a global hypothesis is defined as a particular history of data associations for all detected targets. The global hypotheses may contain Bernoulli components with uncertain existence, i.e., $r < 1$. From each global hypothesis with uncertain target existences, global hypotheses with certain target existence can easily be found. In general, an MB process with s components with uncertain existence can be represented by 2^s MB processes with certain existence. For the example with two consecutive measurement sets, both with two measurements, there are 2 global hypotheses after the first time step, and 15 global hypotheses after the second time step. With certain global hypotheses, the corresponding numbers are

⁴The sequence of Bell numbers is logarithmically convex, i.e., $B(n)^2 \leq B(n-1)B(n+1)$ for $n \geq 1$ [18]. If the Bell numbers are divided by the factorials, $\frac{B(n)}{n!}$, the sequence is logarithmically concave, $\left(\frac{B(n)}{n!}\right)^2 \geq \frac{B(n-1)}{(n-1)!} \frac{B(n+1)}{(n+1)!}$, for $n \geq 1$ [19].

5 hypotheses after the first time step, and 57 hypotheses after the second time step. We conclude that a MBM with uncertain target existence is more compact compared to a MBM with certain target existence.

B. Approximations of the data association problem

To achieve computational tractability, it is necessary to reduce the number of PMBM parameters. Here we will briefly describe the strategy for doing this that was used to obtain the results presented in Section VII. First, the number of data associations is reduced using gating, clustering, and ranking of the association events. Second, after an updated PMBM has been computed, we reduce the number of parameters using pruning, merging, and recycling.

1) *Reducing the number of associations:* First, gating, described in, e.g., [20, Sec. 2.2.2.2], is performed; naturally the extended target gates take into account both the position and the extent of the target, as well as state uncertainties. Given the gating, the targets and the measurements are separated into approximately independent sub-groups, using a method similar to the one proposed in [21, Sec. 3]. After the grouping, we use the methods proposed in [7], [8] to compute several different partitionings of the measurements. Lastly, for each partitioning we compute the M best assignments using Murty's algorithm [22]. This three step procedure—gating, partitioning, assignment—results in a subset of associations $\hat{\mathcal{A}} \subseteq \mathcal{A}$, and typically reduces the number of associations in the update by several orders of magnitude. Similar approaches to reducing the number of data associations have been used previously in several extended target tracking filters, see [6]–[9].

2) *Reducing the number of parameters:* After the PMBM update, MBM components whose updated weight fall below a threshold are pruned from the MBM. For the remaining MBM components, we apply the recycling method suggested in [23], [24]. All Bernoullis with probability of existence below a threshold τ_{rec} are removed from the MBM, approximated as a PPP with intensity $rf(\mathbf{x})$, and this intensity is added to the updated undetected PPP density. If the PPP intensity is represented by a distribution mixture, which is the typical case, then similar mixture components can be merged, e.g., by minimising the KL-div, and mixture components with low weights can be pruned from the PPP intensity. Lastly, we apply the merging algorithm outlined in Section V-C to the MBM.

C. Multi-Bernoulli mixture merging

In [25] an approximate Poisson Multi-Bernoulli filter for point target tracking is proposed, where the PMBM density that results after the update is approximated as a PMB density by using variational approximation to minimise the Kullback-Leibler divergence (KL-div) between the true PMBM density and the approximate PMB density. Empirically, we have found that in extended object filtering it is generally not advisable to merge the whole PMBM density to a single PMB density. The main reason is the extent state: merging two densities with significantly different extent estimates will result in an approximate density in which the extent estimates are distorted.

However, in extended target tracking, similar components in the PMBM density can be merged, in order to reduce the computational cost of the PMBM filter.

Consider an MBM density with MB components indexed by the index set \mathbb{J} . The KL-div between two multi-Bernoulli densities $j_1 \in \mathbb{J}$ and $j_2 \in \mathbb{J}$, with equal number of Bernoulli components $|\mathbb{J}^{j_1}| = |\mathbb{J}^{j_2}|$, is upper bounded [25]

$$D(f^{j_1} || f^{j_2}) \leq \sum_{\pi \in \Pi} q(\pi) \prod_{i \in \mathbb{J}^{j_1}} \int f^{j_1, i}(\mathbf{X}^i) \log \left(q(\pi) \frac{f^{j_1, i}(\mathbf{X}^i)}{f^{j_2, \pi(i)}(\mathbf{X}^i)} \right) \delta \mathbf{X}^i \quad (19)$$

where Π is the set of all ways to assign the Bernoulli components indexed by \mathbb{J}^{j_1} to the Bernoulli components indexed by \mathbb{J}^{j_2} , and $q(\pi) \in [0, 1]$ are weights for the assignments π , $\sum_{\pi \in \Pi} q(\pi) = 1$.

For two MB densities, we compute the pairwise KL-div between the Bernoulli densities, and compute an assignment $\hat{\pi}$ that gives the minimal sum of KL-div. Setting $q(\hat{\pi}) = 1$ we get

$$D(f^{j_1} || f^{j_2}) \leq \prod_{i \in \mathbb{J}^{j_1}} \int f^{j_1, i}(\mathbf{X}^i) \log \left(\frac{f^{j_1, i}(\mathbf{X}^i)}{f^{j_2, \hat{\pi}(i)}(\mathbf{X}^i)} \right) \delta \mathbf{X}^i = D_{UB}(f^{j_1} || f^{j_2}) \quad (20)$$

where the subscript UB denotes the upper bound. In this work, we use MBM merging and merge MB densities for which $D_{UB}(f^{j_1} || f^{j_2})$ is smaller than a threshold.

D. Approximation error

The PMBM density (10) can be rewritten as a mixture of Poisson Multi-Bernoulli densities,

$$f(\mathbf{X}) = \sum_{j \in \mathbb{J}} \mathcal{W}^j \sum_{\mathbf{X}^u \cup \mathbf{X}^d = \mathbf{X}} f^u(\mathbf{X}^u) f^j(\mathbf{X}^d), \quad (22)$$

where the Poisson density $f^u(\mathbf{X}^u)$ is equal for all components. Using gating, partitioning, and assignment, we seek to prune low weight components from the mixture density, such that only components with significant weights remain. Trivially, pruning updated MBM components with low weights would achieve precisely this. Let

$$f_{\mathbb{J}}(\mathbf{X}) = \sum_{j \in \mathbb{J}} \mathcal{W}^j f^j(\mathbf{X}), \quad f_{\mathbb{H}}(\mathbf{X}) = \sum_{j \in \mathbb{H}} \mathcal{W}^j f^j(\mathbf{X}), \quad (23)$$

be two unnormalized PMBM densities with non-negative weights (i.e., the weights do not necessarily sum to one). If $\mathbb{H} \subseteq \mathbb{J}$, then [26, Prop. 5] shows that the L_1 -error incurred when approximating $f_{\mathbb{J}}(\mathbf{X})$ with $f_{\mathbb{H}}(\mathbf{X})$ satisfies

$$\|f_{\mathbb{J}} - f_{\mathbb{H}}\|_1 = \sum_{j \in \mathbb{J} \setminus \mathbb{H}} \mathcal{W}_j, \quad (24a)$$

$$\left\| \frac{f_{\mathbb{J}}}{\|f_{\mathbb{J}}\|_1} - \frac{f_{\mathbb{H}}}{\|f_{\mathbb{H}}\|_1} \right\|_1 \leq 2 \frac{\|f_{\mathbb{J}}\|_1 - \|f_{\mathbb{H}}\|_1}{\|f_{\mathbb{J}}\|_1}. \quad (24b)$$

This result supports the intuitive idea that keeping components with large weights, and discarding components with minimal weights, will yield a small L_1 -error. Further, this shows us that

TABLE II
GGIW UPDATE

Input: GGIW parameter ζ_+ , set of detections \mathbf{W} , measurement model H .
Output: Updated GGIW parameter ζ and predicted likelihood ℓ :

$$\zeta = \begin{cases} \alpha &= \alpha_+ + |\mathbf{W}|, \\ \beta &= \beta_+ + 1, \\ \mathbf{m} &= \mathbf{m}_+ + K\varepsilon, \\ P &= P_+ - KHP_+, \\ v &= v_+ + |\mathbf{W}|, \\ V &= V_+ + N + Z \end{cases}$$

where

$$\begin{aligned} \bar{\mathbf{z}} &= \frac{1}{|\mathbf{W}|} \sum_{\mathbf{z}^i \in \mathbf{W}} \mathbf{z}^i, \\ Z &= \sum_{\mathbf{z}^i \in \mathbf{W}} (\mathbf{z}^i - \bar{\mathbf{z}})(\mathbf{z}^i - \bar{\mathbf{z}})^T \\ \hat{X} &= V_+ (v_+ - 2d - 2)^{-1}, \\ \varepsilon &= \bar{\mathbf{z}} - H\mathbf{m}_+, \\ S &= HP_+H^T + \frac{\hat{X}}{|\mathbf{W}|}, \\ K &= P_+H^T(S)^{-1}, \\ N &= \hat{X}^{1/2}S^{-1/2}\varepsilon\varepsilon^TS^{-T/2}\hat{X}^{T/2} \end{aligned}$$

Predicted likelihood, where $\Gamma(\cdot)$ is the Gamma function, and $\Gamma_d(\cdot)$ is the multivariate Gamma function,

$$\ell = \left(\pi^{|\mathbf{W}|} |\mathbf{W}| \right)^{-\frac{d}{2}} \frac{|V_+|^{\frac{v_+ - d - 1}{2}} \Gamma_d\left(\frac{v_+ - d - 1}{2}\right) |\hat{X}|^{\frac{1}{2}} \Gamma(\alpha) (\beta_+)^{\alpha_+}}{|V|^{\frac{v - d - 1}{2}} \Gamma_d\left(\frac{v - d - 1}{2}\right) |S|^{\frac{1}{2}} \Gamma(\alpha_+) (\beta)^\alpha}$$

it is possible to achieve an arbitrarily accurate approximation by keeping more components, which in turn shows us that conjugate priors based on MB densities may be useful even though the theoretical growth of the number of components is hyperexponential. After pruning PMBM components, the approximate density is normalised. Assuming that $f_{\mathbb{J}}$ is normalised and its approximation $f_{\mathbb{H}}$ is not, (24b) shows that the L_1 -error for the normalized approximation is less than two times the sum of the truncated weights.

In [13] it is shown that the minimum Kullback-Liebler divergence PPP approximation of a Bernoulli density is a PPP whose intensity is equal to the product of the Bernoulli existence probability and state density. In other words, the recycling in Section V-B2 minimises the KL-div. Setting the recycling threshold $\tau_{rec} = 0.1$ is suggested in [23], [24], where it is shown to give small KL-div errors. Similarly, by choosing a low threshold in the MB merging algorithm, we guarantee that the resulting approximation error has low error. Lastly, using a reasoning similar to (24), it can be shown that pruning the PPP intensity by removing low weight components, and merging similar components by minimising the KL-div, incurs a small error.

VI. GGIW IMPLEMENTATION

In this section, an implementation of the PMBM filter is presented. There are several single extended target models available in the literature, see [1] for an overview. Here we have chosen the random matrix model [27], [28], in which the target shape is approximated by an ellipse. The random matrix model is relatively simple to use, yet flexible enough to be applicable to data from radar, lidar, and camera. Furthermore, it has been used in many other multiple extended target tracking filters, making comparison easy. A comprehensive

TABLE III
GGIW PREDICTION

Input: GGIW parameter ζ , motion model $g(\cdot)$, process noise covariance Q , transformation matrix $M(\cdot)$, sampling time T_s , maneuvering correlation constant τ , measurement rate parameter η .

Output: Predicted GGIW parameter ζ_+ , where $G = \nabla_{\xi} g(\xi)|_{\xi=\mathbf{m}}$,

$$\zeta_+ = \begin{cases} \alpha_+ &= \frac{\alpha}{\eta}, \\ \beta_+ &= \frac{\beta}{\eta}, \\ \mathbf{m}_+ &= g(\mathbf{m}), \\ P_+ &= GPG^T + Q, \\ v_+ &= 2d + 2 + e^{-T_s/\tau} (v - 2d - 2), \\ V_+ &= e^{-T_s/\tau} M(\mathbf{m})VM(\mathbf{m})^T \end{cases}$$

discussion of the random matrix model is given in [1, Sec. 3]

A. Single target models

In the random matrix model, the extended target state \mathbf{x}_k is the combination of the scalar γ_k , the vector ξ_k and the matrix X_k . The random vector $\xi_k \in \mathbb{R}^{n_x}$ is the kinematic state, which describes the target's position and its motion parameters (e.g., velocity, acceleration and turn-rate). The random matrix $X_k \in \mathbb{S}_{++}^d$ is the extent state and describes the target's size and shape, and d is the dimension of the extent (typically $d = 2$ or $d = 3$). Lastly, the random variable $\gamma_k > 0$ is the measurement model Poisson rate.

The measurement likelihood for a single measurement \mathbf{z} , cf. (5), is

$$\phi(\mathbf{z}_k | \mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k; H_k \xi_k, X_k), \quad (25)$$

where H_k is a known measurement model. The single-target conjugate prior for the PPP model (5) with single measurement likelihood (25) is a GGIW distribution [28], [29],

$$f_{k|k}(\mathbf{x}) = \mathcal{G}(\gamma_k; \alpha_{k|k}, \beta_{k|k}) \mathcal{N}(\xi_k; \mathbf{m}_{k|k}, P_{k|k}) \times \mathcal{IW}_d(X_k; v_{k|k}, V_{k|k}), \quad (26)$$

$$= \mathcal{GGIW}(\mathbf{x}_k; \zeta_{k|k}), \quad (27)$$

where $\zeta_{k|k} = \{\alpha_{k|k}, \beta_{k|k}, m_{k|k}, P_{k|k}, v_{k|k}, V_{k|k}\}$ is the set of GGIW density parameters. The gamma distribution is the conjugate prior for the unknown Poisson rate, and the Gaussian-inverse Wishart distributions are the conjugate priors for Gaussian distributed detections with unknown mean and covariance.

For a GGIW distribution with prior parameters $\zeta_{k|k-1}$, that is updated with a set of detections \mathbf{W} under the linear Gaussian model (25), the updated parameters $\zeta_{k|k}$, and the corresponding predicted likelihood, are given in Table II. For further discussions about the measurement update within the random matrix extended target model see, e.g., [27], [28], [30].

The motion models are

$$\xi_{k+1} = g(\xi_k) + \mathbf{w}_k, \quad (28a)$$

$$X_{k+1} = M(\xi_k)X_kM(\xi_k)^T, \quad (28b)$$

$$\gamma_{k+1} = \gamma_k. \quad (28c)$$

where $g(\cdot)$ is a kinematic motion model, \mathbf{w}_k is Gaussian process noise with zero mean and covariance Q , and $M(\xi_k)$ is a

TABLE IV
ASSUMPTIONS

- GGIW birth PPP intensity with known parameters,

$$D_{k+1}^b(\mathbf{x}) = \sum_{n=1}^{N_{k+1}^b} w_{k+1}^{b,n} \mathcal{GGIW}(\mathbf{x}_{k+1}; \zeta_{k+1}^{b,n}) \cdot t \quad (29)$$

- GGIW initial undetected PPP intensity with known parameters,

$$D_0^u(\mathbf{x}) = \sum_{n=1}^{N_0^u} w_0^{u,n} \mathcal{GGIW}(\mathbf{x}_0; \zeta_0^{u,n}) \cdot \quad (30)$$

- Empty initial MBM: $\mathbb{J}_0 = \{j_1\}$, $\mathcal{W}_0^{j_1} = 1$, and $\mathbb{I}_0^{j_1} = \emptyset$.
- Probabilities of detection and survival can be approximated as

$$p_D(\mathbf{x})f(\mathbf{x}) \approx p_D(\hat{\mathbf{x}})f(\mathbf{x}), \quad p_S(\mathbf{x})f(\mathbf{x}) \approx p_S(\hat{\mathbf{x}})f(\mathbf{x}). \quad (31)$$

where $\hat{\mathbf{x}} = \mathbb{E}[\mathbf{x}] = \int \mathbf{x}f(\mathbf{x})d\mathbf{x}$.

- Clutter Poisson rate λ is known and the spatial distribution is uniform, $c(\mathbf{z}) = A^{-1}$, where A is the volume of the surveillance region.

TABLE V
GGIW PMBM PREDICTION

Input: $D^u, \{(\mathcal{W}^j, \{(r^{j,i}, f^{j,i})\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}$.

Output: $D_+^u, \{(\mathcal{W}_+^j, \{(r_+^{j,i}, f_+^{j,i})\}_{i \in \mathbb{I}_+^j})\}_{j \in \mathbb{J}}$

$$\begin{aligned} D_+^u(\mathbf{x}) &= \sum_{n=1}^{N^b} w^{b,n} \mathcal{GGIW}(\mathbf{x}; \zeta^{b,n}) \\ &+ \sum_{n=1}^{N^u} w^{u,n} p_S(\hat{\mathbf{x}}^{u,n}) \mathcal{GGIW}(\mathbf{x}; \zeta_+^{u,n}) \\ r_+^{j,i} &= p_S(\hat{\mathbf{x}}^{j,i}) r^{j,i} \\ f_+^{j,i}(\mathbf{x}) &= \mathcal{GGIW}(\mathbf{x}; \zeta_+^{j,i}) \end{aligned}$$

and $\mathcal{W}_+^j = \mathcal{W}^j$, where $\zeta_+^{u,n}$ and $\zeta_+^{j,i}$ are computed as in Table III.

transformation matrix. For these motion models, the predicted parameters $\zeta_{k+1|k}$ for a GGIW distribution with posterior parameters $\zeta_{k|k}$ are given in Table III. For longer discussions about prediction within the random matrix extended target model, see, e.g., [27], [28], [31].

B. Pseudo code for the update and the prediction

The GGIW-PMBM filter propagates in time the GGIW-PMBM density parameters, using a recursion that consists of an update and a prediction. The assumptions are listed in Table IV. The assumptions about the probabilities of detection and survival hold trivially if $p_D(\cdot)$ and $p_S(\cdot)$ are constants, and the assumptions are expected to hold when $p_D(\cdot)$ and $p_S(\cdot)$ are sufficiently smooth functions within the uncertainty area of the estimate. Note that the assumptions of GGIW mixture intensities for the birth PPP and the initial undetected PPP result in all single target densities in the PMBM filter being GGIW densities, due to the conjugacy property.

The predicted GGIW-PMBM parameters are presented in Table V. The updated density for the undetected PPP has $N_{k+1}^b + N_{k|k}^u$ GGIW components after the prediction, whereas the number of MBM parameters remains the same as before the prediction. The pseudo-code for the PMBM update, under assumed GGIW models, is given in Table VI. This builds upon

TABLE VI
GGIW-PMBM UPDATE

Input: Predicted parameters $D_+^u, \{(\mathcal{W}_+^j, \{(r_+^{j,i}, f_+^{j,i})\}_{i \in \mathbb{I}_+^j})\}_{j \in \mathbb{J}_+}$, measurement set \mathbf{Z}

Output: Updated parameters $D^u, \{(\mathcal{W}^j, \{(r^{j,i}, f^{j,i})\}_{i \in \mathbb{I}^j})\}_{j \in \mathbb{J}}$.

Compute D^u as in Table VII

Initialise: $\mathbb{J} \leftarrow \emptyset, j \leftarrow 0$

for $j_+ \in \mathbb{J}_+$ **do**

 Compute subset of associations $\hat{\mathcal{A}}^{j_+}$

for $A \in \hat{\mathcal{A}}^{j_+}$ **do**

 Increment: $j \leftarrow j + 1, \mathbb{J} \leftarrow \mathbb{J} \cup j$

 Initialise: $\mathbb{I}^j \leftarrow \emptyset, i \leftarrow 0, \mathbb{D} \leftarrow \emptyset, \mathcal{L}_A^{j+} \leftarrow 1$

for $C \in A$ **do**

 Increment: $i \leftarrow i + 1, \mathbb{I}^j \leftarrow \mathbb{I}^j \cup i$

if $C \cap \mathbb{I}^{j+} = \emptyset$ **then**

 From r^{j+,i_C}, f^{j+,i_C} , compute r, f, \mathcal{L} as in Table VIII

else

 From r^{j+,i_C}, f^{j+,i_C} , compute r, f, \mathcal{L} as in Table IX

$\mathbb{D} \leftarrow \mathbb{D} \cup i_C$

end if

$r^{j,i} \leftarrow r, f^{j,i} \leftarrow f, \mathcal{L}_A^{j+} \leftarrow \mathcal{L}_A^{j+} \times \mathcal{L}$

end for

for $i_+ \in (\mathbb{I}^{j+} \setminus \mathbb{D})$ **do**

 Increment: $i \leftarrow i + 1, \mathbb{I}^j \leftarrow \mathbb{I}^j \cup i$

 From r^{j+,i_C}, f^{j+,i_C} , compute r, f, \mathcal{L} as in Table X

$r^{j,i} \leftarrow r, f^{j,i} \leftarrow f, \mathcal{L}_A^{j+} \leftarrow \mathcal{L}_A^{j+} \times \mathcal{L}$

end for

$\mathcal{W}^j \leftarrow \mathcal{L}_A^{j+}$

end for

end for

$$\mathcal{W}^j \leftarrow \frac{\mathcal{W}^j}{\sum_{j' \in \mathbb{J}} \mathcal{W}^{j'}}$$

the PPP intensity updates for missed detection and detection, see Tables VII and VIII, respectively, and the Bernoulli updates for detection and missed detection, see Tables IX and X, respectively.

The density for a target detected for the first time, see Table VII, is multimodal, with one mode for each of the GGIW components in the predicted PPP intensity D^u . Mixture reduction can be used to reduce this to a uni-modal GGIW density [29], [32]. This reduction typically has low error, because one of the modes in the predicted PPP intensity is typically much likelier than the other modes.

The density for a previously detected target that is now missed, see Table X, is multi-modal with two modes. This is due to the fact that there are two ways for the target detection to result in an empty measurement set. The first corresponds to the detection process modeled by $p_D(\cdot)$, which may result in a missed detection. The second corresponds to the Poisson number of detections governed by the parameter γ , i.e., the Poisson random number of detections is zero. Note that the Gaussian and inverse Wishart parameters are identical in both cases, it is only the gamma parameters that differ. Using gamma mixture reduction [29], the bi-modality of the γ_k estimate can be reduced to a single mode.

VII. RESULTS

In this section the results from a Monte Carlo simulation study, and experiments with laser range data, are presented. A comparison between the PHD, CPHD, LMB and δ -GLMB filters is presented in [9]. It shows that the LMB filter and the δ -GLMB filter outperform the PHD filter and the CPHD filter. Therefore,

TABLE VII
GGIW PPP UPDATE: MISSED DETECTION

Input: Predicted PPP intensity $D_+^u(\mathbf{x})$, probability of missed detection $q_D(\mathbf{x})$.
Output: Updated PPP intensity $D^u(\mathbf{x})$:

$$D^u(\mathbf{x}) = \sum_{n=1}^{N^u} (w_1^{u,n} \mathcal{G}\mathcal{G}\mathcal{I}\mathcal{W}(\mathbf{x}; \zeta_1^{u,n}) + w_2^{u,n} \mathcal{G}\mathcal{G}\mathcal{I}\mathcal{W}(\mathbf{x}; \zeta_2^{u,n}))$$

where

$$\begin{aligned} w_1^{u,n} &= (1 - p_D(\hat{\mathbf{x}}^{u,n})) w^{u,n} \\ w_2^{u,n} &= p_D(\hat{\mathbf{x}}^{u,n}) \left(\frac{\beta_+^{u,n}}{\beta_+^{u,n} + 1} \right)^{\alpha_+^{u,n}} w^{u,n} \\ \zeta_1^{u,n} &= \zeta_+^{u,n} \\ \zeta_2^{u,n} &= \{\alpha_+^{u,n}, \beta_+^{u,n} + 1, \mathbf{m}_+^{u,n}, P_+^{u,n}, v_+^{u,n}, V_+^{u,n}\} \end{aligned}$$

TABLE VIII
GGIW PPP UPDATE: DETECTION

Input: Predicted PPP intensity $D_+^u(\mathbf{x})$, and measurements set \mathbf{C} .
Output: Bernoulli parameters r_C^u , $f_C^u(\mathbf{x})$ and likelihood \mathcal{L}_C^u :

$$\begin{aligned} r_C^u &= \begin{cases} 1 & \text{if } |\mathbf{C}| > 1 \\ \frac{\mathcal{L}_C}{\kappa_C + \mathcal{L}_C} & \text{if } |\mathbf{C}| = 1 \end{cases} \\ f_C^u(\mathbf{x}) &= \frac{\sum_{n=1}^{N^u} w^{u,n} p_D(\hat{\mathbf{x}}^{u,n}) \ell_C^{u,n} \mathcal{G}\mathcal{G}\mathcal{I}\mathcal{W}(\mathbf{x}; \zeta_C^{u,n})}{\sum_{n=1}^{N^u} w^{u,n} p_D(\hat{\mathbf{x}}^{u,n}) \ell_C^{u,n}} \\ \mathcal{L}_C^u &= \sum_{n=1}^{N^u} w^{u,n} p_D(\hat{\mathbf{x}}^{u,n}) \ell_C^{u,n} \end{aligned}$$

where $\zeta_C^{u,n}$ and $\ell_C^{u,n}$ are computed as in Table II.

in the simulation study presented here, we focus on comparing the PMBM filter to the δ -GLMB filter and the LMB filter. In [9] two variants of the LMB filter are presented, one with known MB birth and one with an adaptive birth process. We found that the LMB filter with adaptive birth process performed better in the simulated scenarios, and have therefore chosen to only present those results. For further details about the δ -GLMB filter and the LMB filter, refer to [9].

The kinematic state is $\xi_k = [\mathbf{p}_k, \mathbf{v}_k]^T \in \mathbb{R}^4$ and describes the target's position $\mathbf{p}_k \in \mathbb{R}^2$ and velocity $\mathbf{v}_k \in \mathbb{R}^2$. The random matrix $X_k \in \mathbb{S}_{++}^2$ is two-dimensional. The motion model $g(\cdot)$ and process noise covariance Q are

$$g(\xi_k) = \begin{bmatrix} \mathbf{I}_2 & T_s \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 \end{bmatrix} \xi_k, \quad Q = \mathbf{G} \sigma_a^2 \mathbf{I}_2 \mathbf{G}^T, \quad \mathbf{G} = \begin{bmatrix} T_s^2 & T_s \\ 2 & \mathbf{I}_2 \end{bmatrix},$$

where T_s is the sampling time and σ_a is the acceleration standard deviation. Because the kinematic state motion model is constant velocity, the extent transformation function M is an identity matrix, $M(\xi_k) = \mathbf{I}_2$.

For GGIW-PMBM, an estimate of the set of targets is obtained by taking the mean vector of all Bernoulli estimates with existence probability larger than 0.5 from the MB component with largest MB weight. For GGIW- δ -GLMB and GGIW-LMB, target extraction is performed analogously, see [9] for details.

For performance evaluation of extended object estimates with ellipsoidal extents, a comparison study has shown that

TABLE IX
GGIW BERNOULLI UPDATE: DETECTION

Input: Predicted Bernoulli parameters $r_+^{j,i}$, $f_+^{j,i}(\mathbf{x})$, and measurement set \mathbf{C} .
Output: Updated Bernoulli parameters $r_C^{j,i}$, $f_C^{j,i}(\mathbf{x})$, and likelihood $\mathcal{L}_C^{j,i}$:

$$\begin{aligned} r_C^{j,i} &= 1 \\ f_C^{j,i}(\mathbf{x}) &= \mathcal{G}\mathcal{G}\mathcal{I}\mathcal{W}(\mathbf{x}; \zeta_C^{j,i}) \\ \mathcal{L}_C^{j,i} &= r_+^{j,i} p_D(\hat{\mathbf{x}}^{j,i}) \ell_C^{j,i} \end{aligned}$$

where $\zeta_C^{j,i}$ and $\ell_C^{j,i}$ are computed as in Table II.

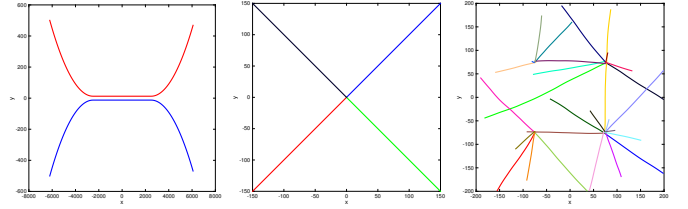


Fig. 1. True target tracks for the three simulated scenarios. In scenario 1 (left), the targets are born well separated, move close to each other, and then split. In scenario 2 (center), the targets are born from the same location, but at different times. In scenario 3 (right), there are four different birth locations.

among six compared performance measures, the Gaussian Wasserstein Distance (GWD) metric is the best choice [33]. The GWD is defined as [34]

$$\begin{aligned} d_{\text{GW}}(\mathbf{x}, \hat{\mathbf{x}}) &= \|H\xi - H\hat{\xi}\|^2 \\ &\quad + \text{Tr} \left(X + \hat{X} - 2 \left(X^{\frac{1}{2}} \hat{X} X^{\frac{1}{2}} \right)^{\frac{1}{2}} \right), \end{aligned} \quad (32)$$

where the measurement model H picks out the position from the state vector. The GWD single target metric is integrated into the Generalised Optimal Sub-Pattern Assignment (GOSPA) multi object metric [35], defined as

$$\begin{aligned} d_p^{(c,2)}(\mathbf{X}, \hat{\mathbf{X}}) &= \left[\min_{\theta \in \Theta(|\mathbf{X}|, |\hat{\mathbf{X}}|)} \sum_{(i,j) \in \theta} d_{\text{GW}}^{(c)}(\mathbf{x}^i, \hat{\mathbf{x}}^j)^p \right. \\ &\quad \left. + \frac{c^p}{2} (|\mathbf{X}| - |\theta| + |\hat{\mathbf{X}}| - |\theta|) \right]^{\frac{1}{p}} \end{aligned} \quad (33)$$

where $d_{\text{GW}}^{(c)}(\mathbf{x}^i, \hat{\mathbf{x}}^j) = \min(c; d_{\text{GW}}(\mathbf{x}^i, \hat{\mathbf{x}}^j))$, $\Theta(|\mathbf{X}|, |\hat{\mathbf{X}}|)$ is the set of all possible 2D assignment sets, c denotes the cut-off at base distance, and p determines the severity of penalizing the outliers in the localisation component. Here we use $c = 10$, $p = 1$.

The GOSPA metric was proposed in [35] as a generalisation of the OSPA metric [36] that allows a decomposition of the error into three parts: 1) localisation error $\sum_{(i,j) \in \theta} d_{\text{GW}}(\mathbf{x}^i, \hat{\mathbf{x}}^j)^p$, 2) missed targets $\frac{c^p}{2} (|\mathbf{X}| - |\theta|)$, and 3) false targets $\frac{c^p}{2} (|\hat{\mathbf{X}}| - |\theta|)$.

A. Simulation study

Three scenarios were simulated; the first two have been used in previous work to evaluate extended target tracking, see [6]–[9], the third was constructed for this paper. For each scenario,

TABLE X
GGIW BERNOULLI UPDATE: MISSED DETECTION

Input: Predicted Bernoulli parameters $r^{j,i}, f^{j,i}(\mathbf{x})$.
Output: Updated Bernoulli parameters $r_\emptyset^{j,i}, f_\emptyset^{j,i}(\mathbf{x})$, and likelihood $\mathcal{L}_\emptyset^{j,i}$:

$$r_\emptyset^{j,i} = \frac{r^{j,i} q_D^{j,i}}{1 - r^{j,i} + r^{j,i} q_D^{j,i}}$$

$$f_\emptyset^{j,i}(\mathbf{x}) = w_1^{j,i} \mathcal{GGIW}(\mathbf{x}_k; \zeta_1^{j,i}) + w_2^{j,i} \mathcal{GGIW}(\mathbf{x}_k; \zeta_2^{j,i})$$

$$\mathcal{L}_\emptyset^{j,i} = 1 - r^{j,i} + r^{j,i} q_D^{j,i}$$

where

$$q_D^{j,i} = 1 - p_D(\hat{\mathbf{x}}^{j,i}) + p_D(\hat{\mathbf{x}}^{j,i}) \left(\frac{\beta^{j,i}}{\beta^{j,i} + 1} \right)^{\alpha^{j,i}}$$

$$w_1^{j,i} = (q_D^{j,i})^{-1} (1 - p_D(\hat{\mathbf{x}}^{j,i}))$$

$$w_2^{j,i} = (q_D^{j,i})^{-1} p_D(\hat{\mathbf{x}}^{j,i}) \left(\frac{\beta^{j,i}}{\beta^{j,i} + 1} \right)^{\alpha^{j,i}}$$

$$\zeta_1^{j,i} = \zeta^{j,i}$$

$$\zeta_2^{j,i} = \{\alpha^{j,i}, \beta^{j,i} + 1, m^{j,i}, P^{j,i}, v^{j,i}, V^{j,i}\}$$

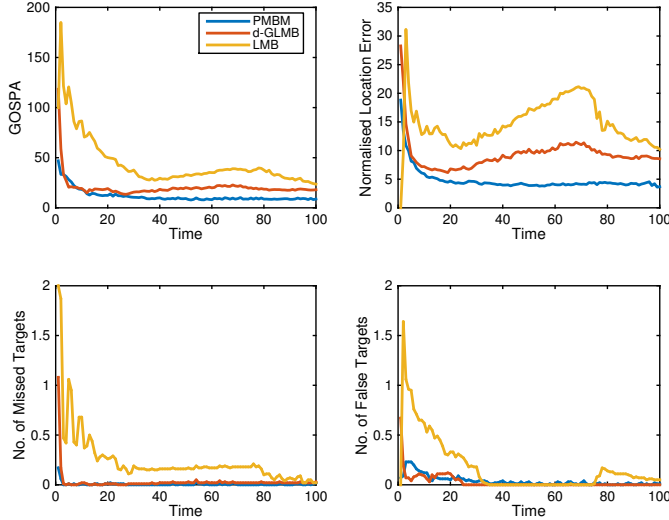


Fig. 2. Results for simulation scenario 1.

100 Monte Carlo runs were performed, and the presented results are averaged over the Monte Carlo runs.

In the first scenario, two targets are simulated for 100 time steps. The true trajectories are shown in Figure 1, and the parameters were set to $p_D = 0.98$, $p_S = 0.99$, and $\lambda = 30$. This scenario is challenging because when the targets are close their measurements form a single cluster, making the data association difficult. The GOSPA performance is shown in Figure 2. For this scenario the total times to process one sequence of measurement sets (mean \pm one standard deviation) were 46 ± 5 s for PMBM, 177 ± 32 s for δ -GLMB, and 7 ± 1 s for LMB.

In the second scenario, four targets were simulated for 200 time steps. The true trajectories are shown in Figure 1, and the parameters were set to $p_D = 0.80$, $p_S = 0.99$, and $\lambda = 30$. The targets appear at different times from the same birth location, and disappear at different times. This scenario illustrates how

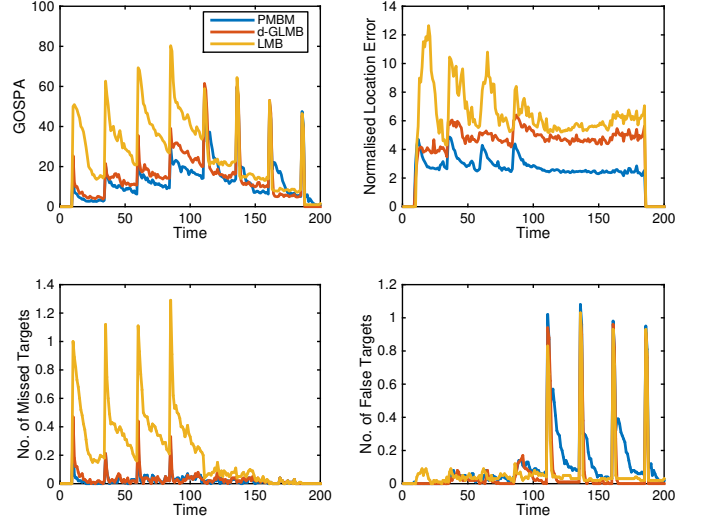


Fig. 3. Results for simulation scenario 2.

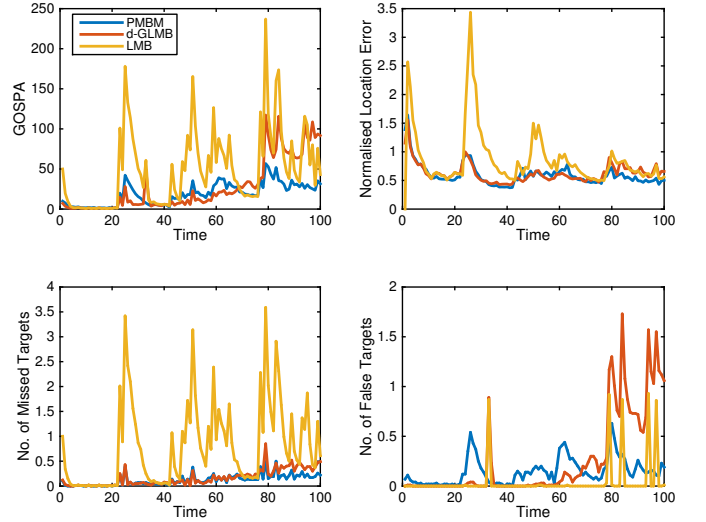


Fig. 4. Results for simulation scenario 3.

the different filters handle target birth and target death. The GOSPA performance is shown in Figure 3. For this scenario the total times to process one sequence of measurement sets (mean \pm one standard deviation) were 32 ± 2 s for PMBM, 287 ± 26 s for δ -GLMB, and 8 ± 4 s for LMB.

In the third scenario, 27 randomly generated targets were simulated for 100 time steps. The true trajectories are shown in Figure 1. The scenario has 100 time steps, and the targets appear in, and disappear from, the surveillance area at different time steps. The parameters were set to $p_D = 0.90$, $p_S = 0.99$, and $\lambda = 60$. The birth spatial density consists of four GGIW components, with positions in $[\pm 75, \pm 75]^T$. This scenario illustrates how the different filters handle a higher target number and higher clutter density. The GOSPA performance is shown in Figure 4. For this scenario the total times to process one sequence of measurement sets (mean \pm one standard deviation) were 89 ± 14 s for PMBM, 1450 ± 117 s for δ -GLMB, and 11 ± 1 s for LMB.

From the results of the simulation study we see that the

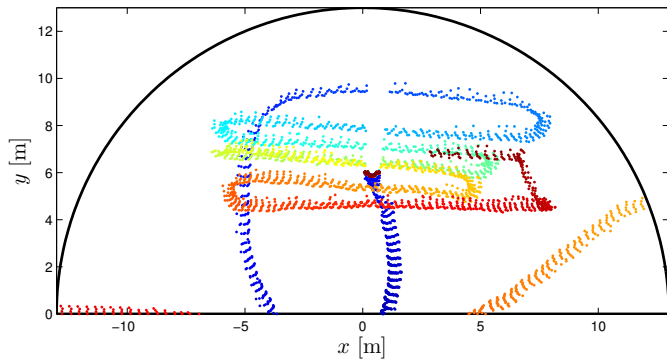


Fig. 5. 2D laser range data, colors are used to visualise different time steps.

GGIW-PMBM filter achieves the lowest GOSPA errors, with GGIW- δ -GLMB second, and GGIW-LMB has the highest GOSPA errors. The computational cost of the PMBM filter is significantly lower than the cost of the δ -GLMB filter, but higher than the cost of the LMB filter. The LMB filter is faster than the PMBM filter because it maintains a single MB density, as opposed to the PMBM which has a mixture of MB densities. However, the single MB density is also why the LMB filter has largest GWD-GOSPA error. That the PMBM is significantly faster than the δ -GLMB filter is mainly due to two reasons: 1) the PMBM has uncertain target existence, whereas the δ -GLMB has certain target existence, and 2) the PMBM is unlabelled which permits merging of similar MB densities. The simulation study shows that for the compared scenarios the PMBM filter achieves an appealing compromise between computational cost and tracking performance.

B. Experiment

An experiment with data from a 2D lidar sensor was performed. This data set has previously been used for tracking using the GGIW-PHD filter [8]. The tracking results for detected targets are essentially identical for this data, since the measurements have relatively low measurement noise and there are very few clutter detections. Instead, the challenges posed by this data, and laser range data in general, are caused by occlusions since a Lidar cannot see behind a target. Because of this we emphasize here the estimation of the PPP density for undetected targets.

The data, shown in Figure 5, contains four pedestrians, two of which remain in the surveillance area for a longer time. One pedestrian moves to the center of the surveillance area and remains there for the remainder of the experiment. The other pedestrian walks around in the surveillance area, both behind and in front of the first pedestrian. A pragmatic approach to handling the occlusions is to use a heterogeneous and time-variant probability of detection $p_D(\mathbf{x})$. Such an approach was used in [8], and it makes it possible to keep track of targets while they are occluded. The method from [8] is used here, and the PPP intensity for undetected targets correctly captures the increased likelihood that a yet undetected target is located in the occluded part of the surveillance area. Results are shown in Figure 6, where the position component of the undetected PPP intensity is shown. The area behind the stationary target is

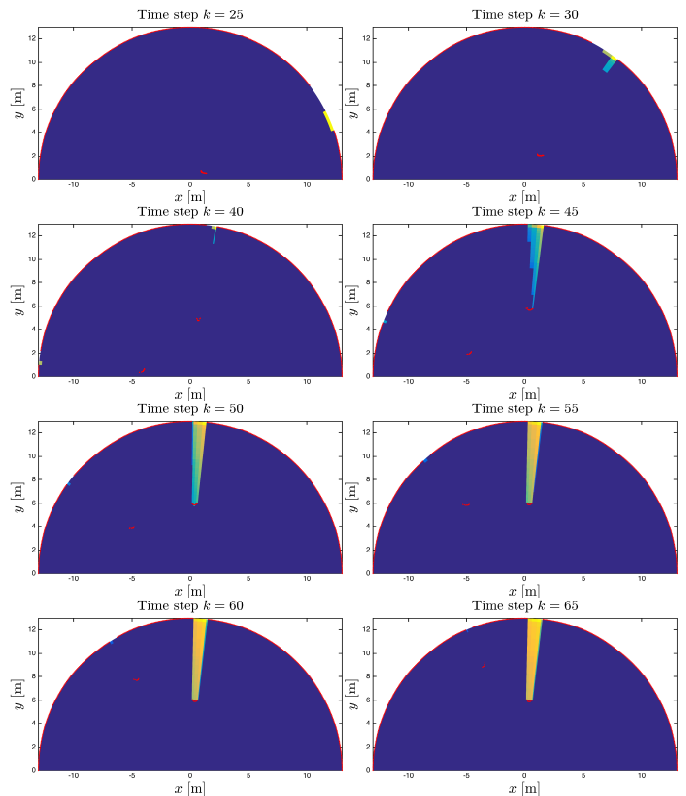


Fig. 6. PMBM tracking results, visualising the position component of the PPP intensity for undetected targets. The red dots are the lidar detections, blue color corresponds to low intensity, green corresponds to intermediate intensity, and yellow high intensity. When the pedestrian remains stationary, the PPP intensity increases behind them.

occluded for an extended period of time, and the PPP intensity correctly captures that in this area we can expect there to be undetected targets. The remaining parts of the surveillance volume, which is not occluded, has very low intensity, which is consistent with our expectation that there is not any undetected targets there.

VIII. CONCLUSIONS

This paper has presented a Poisson multi-Bernoulli mixture conjugate prior for tracking of multiple extended targets. Due to the unknown data associations, the complexity of the update is prohibitive, however, standard MTT methods such as gating and clustering can easily be used to handle this. A GGIW implementation is also presented, for tracking of extended targets with elliptic shapes. A simulation study shows the improved performance compared to the extended target δ -GLMB and LMB filters.

An experiment with laser range data illustrated how the Poisson process helps us to model undetected targets. That is, by approximating the probability of detection, the tracking filter can both track detected targets during occlusions, and represent parts of the surveillance area where yet undetected targets may be located. There are many more scenarios where the probability of detection varies in both time and space, creating a need to model undetected targets. Examples include sensors that scan the surveillance area in a non-deterministic

way, such as radars with narrow lobes that can be focused on certain bearings, or optical sensors mounted on airborne vehicles.

Lastly, please note that labels can be used to form target trajectories from the output of the LMB and δ -GLMB filters. Using the PMBM filter output to form target trajectories is a topic for future work.

APPENDIX

In this appendix we prove Theorem 2 using the probability generating transform (pgfl).

A. PGFL background

In this subsection we give a brief background on the pgfl. Let \mathbf{X} be an RFS with multi-object density $f(\mathbf{X})$. The probability generating functional (pgfl) is a multitarget integral transform of the multi-object density. The pgfl and its inverse are defined as [3]

$$G[h] = \int h^{\mathbf{X}} f(\mathbf{X}) \delta \mathbf{X}, \quad (34a)$$

$$f(\mathbf{X}) = \frac{\delta}{\delta \mathbf{X}} G[h] \Big|_{h=0}, \quad (34b)$$

where $h(\mathbf{x})$ is a test-function. The PPP, Bernoulli, and multi Bernoulli pgfls, corresponding to the densities (2), (3), and (4), respectively, are given by [3]

$$G^{\text{PPP}}[h] = \exp(\langle D; h \rangle - \langle D; 1 \rangle), \quad (35)$$

$$G^{\text{Ber}}[h] = 1 - r + r \langle f; h \rangle, \quad (36)$$

$$G^{\text{MB}}[h] = \prod_{i \in \mathbb{I}} (1 - r^i + r^i \langle f^i; h \rangle). \quad (37)$$

It follows from (34a) that the MBM pgfl is

$$G^{\text{MBM}}[h] = \sum_{j \in \mathbb{J}} \mathcal{W}^j \prod_{i \in \mathbb{I}^j} (1 - r^{j,i} + r^{j,i} \langle f^{j,i}; h \rangle). \quad (38)$$

Due to the standard independence assumption, see Section III-A2, the pgfl of the PMBM density (10) is given by

$$G[h] = G^u[h] \sum_{j \in \mathbb{J}} \mathcal{W}^j G^j[h], \quad (39a)$$

$$G^u[h] = \exp(\langle D^u; h \rangle - \langle D^u; 1 \rangle), \quad (39b)$$

$$G^j[h] = \prod_{i \in \mathbb{I}^j} (1 - r^{j,i} + r^{j,i} \langle f^{j,i}; h \rangle). \quad (39c)$$

We rewrite this as

$$G[h] = \sum_{j \in \mathbb{J}} \mathcal{W}^j G^u[h] \prod_{i \in \mathbb{I}^j} G^{j,i}[h], \quad (40a)$$

$$G^{j,i}[h] = 1 - r^{j,i} + r^{j,i} \langle f^{j,i}; h \rangle. \quad (40b)$$

B. PGFL form of Bayes update

In this subsection we present the Bayes update on pgfl form. Let $G_+[h]$ be the prior pgfl that corresponds to the multi-target density $f_+(\mathbf{X}) = f_{k|k-1}(\mathbf{X}_k | \mathbf{Z}^{k-1})$ in (1a). The pgfl for the

measurement model (Section III-B) with PPP clutter and PPP target measurements is

$$G[g|\mathbf{X}] = G^C[g] (1 - p_D + p_D \exp(\langle \gamma \phi; g - 1 \rangle))^{\mathbf{X}}, \quad (41a)$$

$$G^C[g] = \exp(\langle \kappa; g - 1 \rangle). \quad (41b)$$

The joint target and measurement pgfl is

$$F[g, h] = \int h^{\mathbf{X}} G[g|\mathbf{X}] f_+(\mathbf{X}) \delta \mathbf{X} \quad (42a)$$

$$= G^C[g] G_+[h] (1 - p_D + p_D \exp(\gamma \langle \phi; g \rangle - \gamma)). \quad (42b)$$

Assuming that the prior pgfl $G_+[h]$ is PMBM (39), and inserting into the joint pgfl (42) gives

$$F[g, h] = \sum_{j \in \mathbb{J}} \mathcal{W}^j F^j[g, h], \quad (43a)$$

$$F^j[g, h] = F^{Cu}[g, h] \prod_{i \in \mathbb{I}^j} F^{j,i}[g, h], \quad (43b)$$

$$F^{Cu}[g, h] = G^C[g] G_+^u[h] (1 - p_D + p_D e^{\gamma \langle \phi; g \rangle - \gamma}), \quad (43c)$$

$$F^{j,i}[g, h] = G_+^{j,i}[h] (1 - p_D + p_D e^{\gamma \langle \phi; g \rangle - \gamma}). \quad (43d)$$

The updated pgfl $G[h]$ that corresponds to the Bayes updated density $f_{k|k}(\mathbf{X}_k | \mathbf{Z}^k)$ in (1b) is given by [3, pp. 530–531, s. 14.8.2]

$$G[h] = \frac{\frac{\delta F[g, h]}{\delta \mathbf{Z}} \Big|_{g=0}}{\frac{\delta F[g', h']}{\delta \mathbf{Z}} \Big|_{g'=0, h'=1}}. \quad (44)$$

C. Preliminaries

In this subsection, we establish some preliminary results that will allow us to obtain the differentiation $\frac{\delta F[g, h]}{\delta \mathbf{Z}}$ that is needed in (44). Trivially, it holds that $\frac{\delta F[g, h]}{\delta \mathbf{Z}} = \sum_{j \in \mathbb{J}} \mathcal{W}^j \frac{\delta F^j[g, h]}{\delta \mathbf{Z}}$. From the product rule, it follows that the differentiation $\frac{\delta F^j[g, h]}{\delta \mathbf{Z}}$ consists of combinations of differentiations of $F^{Cu}[g, h]$ and of $F^{j,i}[g, h]$.

Lemma 1: The differentiation of $F^{Cu}[g, h]$ w.r.t. a single measurement \mathbf{z} is

$$\frac{\delta F^{Cu}[g, h]}{\delta \mathbf{z}} = \left(\kappa(\mathbf{z}) + \langle h; D_+^u \ell_{\mathbf{z}} e^{\gamma \langle \phi; g \rangle} \rangle \right) F^{Cu}[g, h], \quad (45)$$

and the differentiation of $\kappa(\mathbf{z}) + \langle h; D_+^u \ell_{\mathbf{z}} e^{\gamma \langle \phi; g \rangle} \rangle$ w.r.t. a set of measurements \mathbf{Y} is

$$\frac{\delta (\kappa(\mathbf{z}) + \langle h; D_+^u \ell_{\mathbf{z}} e^{\gamma \langle \phi; g \rangle} \rangle)}{\delta \mathbf{Y}} = \langle h; D_+^u \ell_{\mathbf{z} \cup \mathbf{Y}} e^{\gamma \langle \phi; g \rangle} \rangle. \quad (46)$$

Lemma 2: The differentiation of $F^{j,i}[g, h]$ w.r.t. a measurement set \mathbf{Z} is

$$\frac{\delta F^{j,i}[g, h]}{\delta \mathbf{Z}} = r_+^{j,i} \langle h; f_+^{j,i} \ell_{\mathbf{Z}} e^{\gamma \langle \phi; g \rangle} \rangle. \quad (47)$$

The proofs of Lemmas 1 and 2 are straightforward and are omitted due to page length constraints.

Lemma 3: The differentiation of $F^j[g, h]$ w.r.t. \mathbf{Z} is

$$\frac{\delta F^j[g, h]}{\delta \mathbf{Z}} = F^{Cu}[g, h] \sum_{A \in \mathcal{A}^j} \prod_{C \in A} F'_C[g, h] \quad (48a)$$

where

$$F'_C[g, h] = \begin{cases} \kappa^{\mathbf{C}_C} + \langle h; D_+^u \ell_{\mathbf{C}_C} e^{\gamma\langle\phi; g\rangle} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| = 1 \\ \langle h; D_+^u \ell_{\mathbf{C}_C} e^{\gamma\langle\phi; g\rangle} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| > 1 \\ F^{j,i_C}[g, h] & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C = \emptyset \\ r_+^{j,i_C} \langle h; f_+^{j,i_C} \ell_{\mathbf{C}_C} e^{\gamma\langle\phi; g\rangle} \rangle & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C \neq \emptyset. \end{cases} \quad (48b)$$

The proof is by induction: For the initial step, assume that $\mathbb{M} = \{m_1\}$. Differentiation, Lemma 1 and Lemma 2, give

$$\frac{\delta F^j[g, h]}{\delta \mathbf{Z}^{m_1}} = F^{Cu}[g, h] \left[\left(\kappa(\mathbf{z}) + \langle h; D_+^u \ell_{\mathbf{z}} e^{\gamma\langle\phi; g\rangle} \rangle \right) \prod_{i \in \mathbb{I}_+^j} F^{j,i}[g, h] + \sum_{i \in \mathbb{I}_+^j} r_+^{j,i} \langle h; f_+^{j,i} \ell_{\mathbf{z}} e^{\gamma\langle\phi; g\rangle} \rangle \prod_{i \in \mathbb{I}_+^j, i \neq i} F^{j,i}[g, h] \right]. \quad (49)$$

We see that (49) is consistent with (48): we have the partitions of $\{m_1\} \cup \mathbb{I}_+^j$, for which there is at most one $i \in \mathbb{I}_+^j$ in each cell. The first row corresponds to a partition in which m_1 is placed in a cell of its own, $\{m_1\}, \{i_1\}, \dots, \{i_I\}$, i.e., an association in which none of the previously detected targets are detected, and the single measurement is either from clutter or a new target. The second row corresponds to partitions $\{i_1\}, \dots, \{m_1, i\}, \dots, \{i_I\}$, i.e., associations where the single measurement is associated to one of the existing targets.

Now, assume that we have established (48) for

$$\mathbb{M}_- = \{m_1, \dots, m_{M-1}\} \quad (50)$$

and that we are to establish (48) for

$$\mathbb{M} = \{m_1, \dots, m_{M-1}, m_M\}. \quad (51)$$

For the sake of notational clarity, let \mathcal{A}_-^j be the association space corresponding to the index set \mathbb{M}_- , and let \mathcal{A}^j be the association space corresponding to the index set \mathbb{M} . Differentiation gives

$$\begin{aligned} & \frac{\delta F^{C,u}[g, h] \sum_{A \in \mathcal{A}_-^j} \prod_{C \in A} F'_C[g, h]}{\delta \mathbf{Z}^{m_M}} \\ &= F^{Cu}[g, h] \left[\sum_{A \in \mathcal{A}_-^j} \sum_{\tilde{C} \in A} \frac{\delta F'_{\tilde{C}}[g, h]}{\delta \mathbf{Z}^{m_M}} \prod_{C \in A, C \neq \tilde{C}} F'_C[g, h] \right. \\ & \quad \left. + \left(\kappa(\mathbf{z}^{m_M}) + \langle h; D_+^u \ell_{\mathbf{z}^{m_M}} e^{\gamma\langle\phi; g\rangle} \rangle \right) \sum_{A \in \mathcal{A}_-^j} \prod_{C \in A} F'_C[g, h] \right]. \end{aligned} \quad (52)$$

The first row corresponds to new association partitions formed by adding the measurement index m_M to one of the existing cells in an association $A \in \mathcal{A}_-^j$, and the second row corresponds to new association partitions formed by putting m_M into a new cell and adding this cell to $A \in \mathcal{A}_-^j$. Together, this constitutes a summation over all possible ways to partition $\mathbb{M} \cup \mathbb{I}_+^j$, i.e., a summation over all associations in the association space \mathcal{A}^j , and (52) is thus consistent with (48). This concludes the proof of Lemma 3.

Lemma 4: For scalar a , scalar b , function c and test function h , the following relation holds

$$a + b \langle h; c \rangle = \mathcal{L}(1 - r + r \langle h; f \rangle), \quad (53a)$$

where

$$\mathcal{L} = a + b \langle c; 1 \rangle, \quad r = \frac{b \langle c; 1 \rangle}{a + b \langle c; 1 \rangle}, \quad f = \frac{c}{\langle c; 1 \rangle}. \quad (53b)$$

The proof is trivial.

D. Proof of update

In this subsection, we show that a prior PMBM pgfl $G_+[h]$ of the form (39) and the measurement model (41a) result in a PMBM pgfl that corresponds to the PMBM density given in Theorem 2. Let the set of measurements \mathbf{Z} be indexed by the index set \mathbb{M} , $\mathbf{Z} = \{\mathbf{z}^m\}_{m \in \mathbb{M}}$. Differentiating, using Lemma 3, and setting $g = 0$, we get

$$\frac{\delta F[g, h]}{\delta \mathbf{Z}} \Big|_{g=0} = F^{C,u}[0, h] \sum_{j \in \mathbb{J}} \mathcal{W}^j \sum_{A \in \mathcal{A}^j} \prod_{C \in A} F'_C[0, h]. \quad (54)$$

where

$$F'_C[0, h] = \begin{cases} \kappa^{\mathbf{C}_C} + \langle h; D_+^u \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| = 1 \\ \langle h; D_+^u \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| > 1 \\ 1 - r_+^{j,i_C} + r_+^{j,i_C} \langle h; f_+^{j,i_C} q_D \rangle & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C = \emptyset \\ r_+^{j,i_C} \langle h; f_+^{j,i_C} \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C \neq \emptyset. \end{cases} \quad (55)$$

Applying Lemma 4, we get

$$\begin{aligned} & \frac{\delta F[g, h]}{\delta \mathbf{Z}} \Big|_{g=0} \\ &= F^{C,u}[0, h] \sum_{j \in \mathbb{J}} \sum_{A \in \mathcal{A}^j} \mathcal{W}^j \prod_{C \in A} \mathcal{L}_C (1 - r_C + r_C \langle h; f_C \rangle) \end{aligned} \quad (56)$$

where

$$\mathcal{L}_C = \begin{cases} \kappa^{\mathbf{C}_C} + \langle D_+^u; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| = 1 \\ \langle D_+^u; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| > 1 \\ 1 - r_+^{j,i_C} + r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C = \emptyset \\ r_+^{j,i_C} \langle f_+^{j,i_C}; \ell_{\mathbf{C}_C} \rangle & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C \neq \emptyset \end{cases} \quad (57a)$$

$$r_C = \begin{cases} \frac{\langle D_+^u; \ell_{\mathbf{C}_C} \rangle}{\kappa^{\mathbf{C}_C} + \langle D_+^u; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| = 1 \\ 1 & \text{if } C \cap \mathbb{I}_+^j = \emptyset, |\mathbf{C}_C| > 1 \\ \frac{r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle}{1 - r_+^{j,i_C} + r_+^{j,i_C} \langle f_+^{j,i_C}; q_D \rangle} & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C = \emptyset \\ 1 & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C \neq \emptyset \end{cases} \quad (57b)$$

$$f_C(\mathbf{x}) = \begin{cases} \frac{D_+^u(\mathbf{x}) \ell_{\mathbf{C}_C}(\mathbf{x})}{\langle D_+^u; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}_+^j = \emptyset \\ \frac{f_+^{j,i_C}(\mathbf{x}) q_D(\mathbf{x})}{\langle f_+^{j,i_C}; q_D \rangle} & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C = \emptyset \\ \frac{f_+^{j,i_C}(\mathbf{x}) \ell_{\mathbf{C}_C}(\mathbf{x})}{\langle f_+^{j,i_C}; \ell_{\mathbf{C}_C} \rangle} & \text{if } C \cap \mathbb{I}_+^j \neq \emptyset, \mathbf{C}_C \neq \emptyset. \end{cases} \quad (57c)$$

Setting $h = 1$ we get

$$\frac{\delta F[g, h]}{\delta \mathbf{Z}} \Big|_{g=0, h=1} = F^{C,u}[0, 1] \sum_{j \in \mathbb{J}} \sum_{A \in \mathcal{A}^j} \mathcal{W}^j \prod_{C \in A} \mathcal{L}_C \quad (58)$$

and taking the ratio of (56) and (58), cf. (44), we get the pgfl of the Bayes updated density,

$$G[h] = \frac{F^{Cu}[0, h]}{F^{Cu}[0, 1]} \times \frac{\sum_{j \in \mathbb{J}} \sum_{A \in \mathcal{A}^j} \mathcal{W}^j \prod_{C \in A} \mathcal{L}_C (1 - r_C + r_C(h; f_C))}{\sum_{j \in \mathbb{J}} \sum_{A \in \mathcal{A}^j} \mathcal{W}^j \prod_{C \in A} \mathcal{L}_C} \quad (59)$$

where the ratio

$$\frac{F^{Cu}[0, h]}{F^{Cu}[0, 1]} = \exp \{ \langle D^u; h \rangle - \langle D^u; 1 \rangle \} \quad (60a)$$

$$D^u(\mathbf{x}) = q_D(\mathbf{x}) D_+^u(\mathbf{x}). \quad (60b)$$

We see that $G[h]$ in (59) is a product of (60a), which is the pgfl of a PPP with intensity (60b), and the pgfl of a MBM with parameters (57). This is consistent with Theorem 2, and concludes the proof.

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